

Transforming Cycle Rewriting into String Rewriting (Extended Version)

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Abstract

We present new techniques to prove termination of cycle rewriting, that is, string rewriting on cycles, which are strings in which the start and end are connected. Our main technique is to transform cycle rewriting into string rewriting and then apply state of the art techniques to prove termination of the string rewrite system. We present three such transformations, and prove for all of them that they are sound and complete. Apart from this transformational approach, we extend the use of matrix interpretations as was studied before. We present several experiments showing that often our new techniques succeed where earlier techniques fail.

1998 ACM Subject Classification F.4.2 Grammars and other rewriting systems

Keywords and phrases rewriting systems, string rewriting, termination

1 Introduction

String rewriting can not only be applied on strings, but also on cycles. Cycles can be seen as strings of which the left end is connected to the right end, by which the string has no left end or right end any more. Applying string rewriting on cycles is briefly called cycle rewriting. Rewriting behavior is strongly influenced by allowing cycles, for instance, in string rewriting the single rule $ab \rightarrow ba$ is terminating, but in cycle rewriting it is not, since the string ab represents the same cycle as ba . From the viewpoint of graph transformation, cycle rewriting is very natural. For instance, in [2] it was shown that if all rules of a graph transformation system are string rewrite rules, termination of the transformation system coincides with termination of the cycle rewrite system, and not with termination of the string rewrite system.

In many areas cycle rewriting is more natural than string rewriting. For instance, the problem of 5 dining philosophers can be expressed as a cycle $FTFTFTFTFT$ where F denotes a fork, and T denotes a thinking philosopher. Writing L for a philosopher who has picked up her left fork, but not her right fork, and E for an eating philosopher, a classical (deadlocking) modeling of the dining philosophers problem (for arbitrary many philosophers) can be expressed by the cycle rewrite system consisting of the rules $FT \rightarrow L$, $LF \rightarrow E$, $E \rightarrow FTF$. As a cycle rewrite system this is clearly not terminating.

So both string rewriting and cycle rewriting provide natural semantics for string rewrite systems, also called semi-Thue systems. Historically, string rewriting got a lot of attention as being a particular case of term rewriting, while cycle rewriting hardly got any attention until recently.

In [18] a first investigation of termination of cycle rewriting was made. Some techniques were presented to prove cycle termination, implemented in a tool `torpacyc`. Further a transformation ϕ was given such that for every string rewriting system (SRS) R , string termination of R holds if and only if cycle termination of $\phi(R)$ holds. As a consequence, cycle termination is undecidable. However, for making use of the strong power of current tools for proving termination of string rewriting in order to prove cycle termination, we need a transformation the other way around: transformations ψ such that for every SRS R , cycle termination of R holds if and only if string termination of $\psi(R)$ holds. The ‘if’ direction in this ‘if and only if’ is called ‘sound’, the ‘only if’ is called complete. A new way to prove cycle termination of an SRS R is to apply a tool for proving termination of string rewriting to $\psi(R)$ for a sound transformation ψ . The main topic of this paper is to investigate such transformations, and to exploit them to prove termination of cycle rewriting.

A similar approach to exploit the power of tools for termination of term rewriting to prove a modified property was used before in [8, 7]. However, there the typical observation was that the complete transformations were complicated, and for non-trivial examples termination of $\psi(R)$ could not be proved by the tools, while for much simpler sound (but incomplete) transformations ψ , termination of $\psi(R)$ could often be proved by the tools. In our current setting this is different: we introduce a transformation `split`, for which we prove that it is sound and complete, but we show that for several systems R for which all approaches from [18] fail, cycle termination of R can be concluded from an automatic termination proof of `split(R)` generated by AProVE [6, 1] or $\mathsf{T}\mathsf{T}_2$ [12, 15].

It can be shown that if strings of size n exist admitting cycle reductions in which for every rule the number of applications of that rule is more than linear in n , then all techniques from [18] fail to prove cycle termination. Nevertheless, in quite simple examples this may occur while cycle termination holds. As an example consider the following.

A number of people are in a circle, and each of them carries a number, represented in binary notation with a bounded number of bits. Each of them may increase his/her number by one, as long as it fits in the bounded number of bits. Apart from that, every person may increase the number of bits of the number of its right neighbor by two. In order to avoid trivial non-termination, the latter is only allowed if the leading bit of the number is 0, and the new leading bit is put to 1, and the other to 0, by which effectively one extra bit is added. We will prove that this process will always terminate by giving an SRS in which all of the above steps can be described by a number of cycle rewrite steps, and prove cycle termination. In order to do so we write P for person, and 0 and 1 for the bits of the binary number. For carry handling we introduce an extra symbol c of which the meaning is a 1 with a carry. Assume for every person its number is stored left from it. So if the number ends in 0, by adding one this last bit 0 is replaced by 1, expressed by the rule $0P \rightarrow 1P$. In case the number ends in 1, a carry should be created, since c represents a 1 with a carry this is expressed by the rule $1P \rightarrow cP$. Next the carry should be processed. In case it is preceded by 0, this 0 should be replaced by 1, while the c is replaced by 0; this is expressed by the rule $0c \rightarrow 10$. In case it is preceded by 1, a new carry should be created while again the old carry is replaced by 0; this is expressed by the rule $1c \rightarrow c0$. In this way adding one to any number in binary notation can be expressed by a number of rewrite steps, as long as no overflow occurs. Finally, we have to add a rule to extend the bit size of the number of the

right neighbor: the leading bit should be 0, while it is replaced by 100: adding two extra bits of which the leading one is 1 and the other is 0. This is expressed by the rule $P0 \rightarrow P100$. Summarizing: we have to prove cycle termination of the SRS consisting of the five rules

$$0P \rightarrow 1P, 1P \rightarrow cP, 0c \rightarrow 10, 1c \rightarrow c0, P0 \rightarrow P100.$$

This is fundamentally impossible by the techniques presented in [18]: by one of the techniques the last rule can be removed, but starting in $0^n P$ a reduction can be made in which all of the remaining four rules are applied an exponential number of times, by which the techniques from [18] fail.

In this paper we give two ways to automatically prove that cycle termination holds for the above example R : $\mathsf{T}\mathsf{T}_2$ succeeds in proving termination of $\mathsf{split}(R)$, and the other is a variant of matrix interpretations for which we show that it proves cycle termination.

The paper is organized as follows. Section 2 recalls the basics of cycle rewriting. The main section Section 3 presents three transformations split , rotate , and shift , and proves soundness and completeness of all of them. In Section 4 the matrix approach is revisited and extended. In Section 5 experiments on implementations of our techniques are reported. We conclude in Section 6.

2 Preliminaries

A *signature* Σ is a finite alphabet of symbols. With Σ^* we denote the set of strings over Σ . With ε we denote the empty string and for $u, v \in \Sigma^*$, we write uv for the concatenation of the strings u and v . With $|u|$ we denote the length of string $u \in \Sigma^*$ and for $a \in \Sigma$ and $n \in \mathbf{N}$, a^n denotes n replications of symbol a , i.e. $a^0 = \varepsilon$ and $a^i = aa^{i-1}$ for $i > 0$.

Given a binary relation \rightarrow , we write \rightarrow^i for i steps, $\rightarrow^{\leq i}$ for at most i steps, $\rightarrow^{< i}$ for at most $i - 1$ steps, \rightarrow^* for the reflexive-transitive closure of \rightarrow , and \rightarrow^+ for the transitive closure of \rightarrow . For binary relations \rightarrow_1 and \rightarrow_2 , we write $\rightarrow_1 \cdot \rightarrow_2$ for the composition of \rightarrow_1 and \rightarrow_2 , i.e. $a \rightarrow_1 \cdot \rightarrow_2 c$ iff there exists a b s.t. $a \rightarrow_1 b$ and $b \rightarrow_2 c$.

A *string rewrite system* (SRS) is a finite set R of rules $\ell \rightarrow r$ where $\ell, r \in \Sigma^*$. The *rewrite relation* $\rightarrow_R \subseteq (\Sigma^* \times \Sigma^*)$ is defined as follows: if $w = ulv \in \Sigma^*$ and $(\ell \rightarrow r) \in R$, then $w \rightarrow_R urv$. The *prefix-rewrite relation* \hookrightarrow_R is defined as: if $w = \ell u \in \Sigma^*$ and $(\ell \rightarrow r) \in R$, then $w \hookrightarrow_R ru$. The *suffix-rewrite relation* \dashrightarrow_R is defined as: if $w = ul \in \Sigma^*$ and $(\ell \rightarrow r) \in R$, then $w \dashrightarrow_R ur$.

A (finite or infinite) sequence of rewrite steps $w_1 \rightarrow_R w_2 \rightarrow_R \dots$ is called a *rewrite sequence* (sometimes also a *reduction* or a *derivation*). An SRS R is *non-terminating* if there exists a string $w \in \Sigma^*$ and an infinite rewrite sequence $w \rightarrow_R w_1 \rightarrow_R w_2 \dots$. Otherwise, R is *terminating*.

We recall the notion of cycle rewriting from [18]. A string can be viewed as a cycle, i.e. the last symbol of the string is connected to the first symbol. To represent cycles by strings, we define the equivalence relation \sim as follows:

$$u \sim v \text{ iff } u = w_1 w_2 \text{ and } v = w_2 w_1 \text{ for some strings } w_1, w_2 \in \Sigma^*$$

With $[u]$ we denote the equivalence class of string u w.r.t. \sim .

The *cycle rewrite relation* $\circrightarrow_R \subseteq (\Sigma/\sim \times \Sigma/\sim)$ of an SRS R is defined as

$$[u] \circrightarrow_R [v] \text{ iff } \exists w \in \Sigma^* : u \sim \ell w, (\ell \rightarrow r) \in R, \text{ and } v \sim r w$$

The cycle rewrite relation \circrightarrow_R is called *non-terminating* iff there exists a string $w \in \Sigma^*$ and an infinite sequence $[w] \circrightarrow_R [w_1] \circrightarrow_R [w_2] \circrightarrow_R \dots$. Otherwise, \circrightarrow_R is called *terminating*.

We recall some known facts about cycle rewriting.

► **Proposition 1** (see [18]). *Let Σ be a signature, R be an SRS, and $u, v \in \Sigma^*$.*

1. *If $u \rightarrow_R v$ then $[u] \circ \rightarrow_R [v]$.*
2. *If $\circ \rightarrow_R$ is terminating, then \rightarrow_R is terminating.*
3. *Termination of \rightarrow_R does not necessarily imply termination of $\circ \rightarrow_R$.*
4. *Termination of $\circ \rightarrow_R$ is undecidable.*
5. *For every SRS R there exists a transformed SRS $\phi(R)$ s.t. the following three properties are equivalent:*
 - *\rightarrow_R is terminating.*
 - *$\rightarrow_{\phi(R)}$ is terminating.*
 - *$\circ \rightarrow_{\phi(R)}$ is terminating.*

For an SRS R , the last property implies that termination of \rightarrow_R can be proved by proving termination of the translated cycle rewrite relation $\circ \rightarrow_{\phi(R)}$. In [18] it was used to show that termination of cycle rewriting is undecidable and for further results on derivational complexity for cycle rewriting.

3 Transforming Cycle Termination into String Termination

Proposition 1 and the involved transformation ϕ , which transforms string rewriting into cycle rewriting, provide a method to prove string termination by proving cycle termination. However, it does not provide a method to prove termination of the cycle rewrite relation $\circ \rightarrow_R$ by proving termination of the string rewrite relations \rightarrow_R or $\rightarrow_{\phi(R)}$. Hence, in this section we develop transformations ψ s.t. termination of $\rightarrow_{\psi(R)}$ implies termination of $\circ \rightarrow_R$. We call such a transformation ψ *sound*. However, there are “useless” sound transformations, for instance, transformations where $\psi(R)$ is always non-terminating. So at least one wants to find sound transformations which permit to prove termination of non-trivial cycle rewrite relations. However, a better transformation should fulfill the stronger property that $\rightarrow_{\psi(R)}$ is terminating if and only if $\circ \rightarrow_R$ is terminating. If termination of $\circ \rightarrow_R$ implies termination of $\rightarrow_{\psi(R)}$, then we say ψ is *complete*. For instance, for a complete transformation, non-termination proofs of $\rightarrow_{\psi(R)}$ also imply non-termination of $\circ \rightarrow_R$. Hence, our goal is to find sound and complete transformations ψ .

We will introduce and discuss three transformations *split*, *rotate*, and *shift* where the most important one is the transformation *split*, since it has the following properties: The transformation is sound and complete, and as our experimental results show, it behaves well in practice when proving termination of cycle rewriting. The other two transformations *rotate* and *shift* are also sound and complete, but rather complex and – as our experimental results show – they do not behave as well as the transformation *split* in practice. We include all three transformations in this paper to document some different approaches to transform cycle rewriting into string rewriting.

While we will analyze the transformation *split* in detail and prove its soundness and completeness, for the other two transformations we briefly list their properties where the corresponding proofs can be found in the appendix

3.1 The Transformation Split

The idea of the transformation *split* is to perform a single cycle rewrite step $[u] \circ \rightarrow_R [v]$ step which uses rule $(\ell \rightarrow r) \in R$, by either applying a string rewrite step $u \rightarrow_R v$ or by splitting the rule $(\ell \rightarrow r)$ into two rules $(\ell_A \rightarrow r_A)$ and $(\ell_B \rightarrow r_B)$, where $\ell = \ell_A \ell_B$ and $r = r_A r_B$. Then a cycle rewrite step can be simulated by a prefix and a subsequent suffix rewrite step:

first apply rule $\ell_B \rightarrow r_B$ to a prefix of u and then apply rule $\ell_A \rightarrow r_A$ to a suffix of the obtained string.

► **Example 2.** Let $R = \{abc \rightarrow bbbb\}$ and $[bcdda] \circ \rightarrow_R [bbddb]$. The rule $abc \rightarrow bbbb$ can be split into the rules $a \rightarrow bb$ and $bc \rightarrow bb$ s.t. $bcdda \xrightarrow{\{bc \rightarrow bb\}} bbdda \xrightarrow{\{a \rightarrow bb\}} bbddb$.

We describe the idea of the transformation **split** more formally. It uses the following observation of cycle rewriting: if $[u] \circ \rightarrow_R [v]$, then $u \sim \ell w$, $(\ell \rightarrow r) \in R$, and $v \sim rw$. From $u \sim \ell w$ follows that $u = u_1 u_2$ and $\ell w = u_2 u_1$ for some u_1, u_2 . We consider the cases for u_2 :

1. If $u_i = \varepsilon$ (for $i = 1$ or $i = 2$), then $u = \ell w$ and $u \xrightarrow{\ell \rightarrow r} rw$ by a prefix string-rewrite step.
2. If ℓ is a prefix of u_2 , i.e. $\ell u'_2 = u_2$, then $w = u'_2 u_1$, $u = u_1 \ell u'_2 \rightarrow_R u_1 r u'_2$, and $u_1 r u'_2 \sim rw$.
3. If u_2 is a proper prefix of ℓ , then there exist ℓ_A, ℓ_B with $\ell = \ell_A \ell_B$ s.t. $u_2 = \ell_A$ and ℓ_B is a true prefix of u_1 , i.e. $u_1 = \ell_B w$ and $u = u_1 u_2 = \ell_B w \ell_A \xrightarrow{\{\ell_B \rightarrow r_B\}} r_B w \ell_A \xrightarrow{\{\ell_A \rightarrow r_A\}} r_B w r_A \sim rw$ if $r_A r_B = r$.

The three cases show that a cycle rewrite step $[u] \circ \rightarrow_{\{\ell \rightarrow r\}} [v]$ can either be performed by applying a string rewrite step $u \rightarrow_{\{\ell \rightarrow r\}} v'$ where $v' \sim v$ (cases 1 and 2) or in case 3 by splitting $\ell \rightarrow r$ into two rules $\ell_A \rightarrow r_A$ and $\ell_B \rightarrow r_B$ such that $u \xrightarrow{\{\ell_B \rightarrow r_B\}} u'$ replaces a prefix of u by r_B and $u' \xrightarrow{\{\ell_A \rightarrow r_A\}} v'$ replaces a suffix of u' by r_A s.t. $v' \sim v$.

For splitting a rule $(\ell \rightarrow r)$ into rules $\ell_A \rightarrow r_A$ and $\ell_B \rightarrow r_B$, we may choose any decomposition of r for r_A and r_B (s.t. $r = r_A r_B$). We will work with $r_A = r$ and $r_B = \varepsilon$.

The above cases for cycle rewriting show that a *sound* transformation of the cycle rewrite relation $\circ \rightarrow_R$ into a string rewrite relation is the SRS which consists of all rules of R and all pairs of rules $\ell_B \rightarrow \varepsilon$ and $\ell_A \rightarrow r$ for all $(\ell \rightarrow r) \in R$ and all ℓ_A, ℓ_B with $|\ell_A| > 0$, $|\ell_B| > 0$, and $\ell = \ell_A \ell_B$. However, this transformation does not ensure that the rules evolved by splitting are used as prefix and suffix rewrite steps only. Indeed, the transformation in this form is useless for nearly all cases, since whenever the right-hand side r of a rule $(\ell \rightarrow r) \in R$ contains a symbol $a \in \Sigma$ which is the first or the last symbol in ℓ , then the transformed SRS is non-terminating. For instance, for $R = \{aa \rightarrow aba\}$ the cycle rewrite relation $\circ \rightarrow_R$ is terminating, while the rule $a \rightarrow aba$ (which would be generated by splitting the left-hand side of the rule) leads to non-termination of the string rewrite relation. Note that this also holds if we choose any other decomposition of the right-hand side. Hence, in our transformation we introduce additional symbols to ensure:

- $\ell_B \rightarrow \varepsilon$ can only be applied to a prefix of the string.
- $\ell_A \rightarrow r$ can only be applied to a suffix of the string.
- If $\ell_B \rightarrow \varepsilon$ is applied to a prefix, then also $\ell_A \rightarrow r$ must be applied, in a synchronized manner (i.e. no other rule $\ell'_B \rightarrow \varepsilon$ or $\ell'_A \rightarrow r'$ can be applied in between).

In detail, we will prepend the fresh symbol **B** to the beginning of the string, and append the fresh symbol **E** to the end of the string. These symbols guarantee, that prefix rewrite steps $\ell u \xrightarrow{(\ell \rightarrow r)} r u$ can be expressed with usual string rewrite rules by replacing the left hand side ℓ with **B** ℓ and analogous for suffix rewrite steps $u \ell \xrightarrow{(\ell \rightarrow r)} u r$ by replacing the left hand side ℓ with ℓ **E**.

Let $(\ell_i \rightarrow r_i)$ be the i^{th} rule of the SRS which is split into two rules $\ell_B \rightarrow \varepsilon$ and $\ell_A \rightarrow r_i$, where $\ell_A \ell_B = \ell_i$. After applying the rule $\ell_B \rightarrow \varepsilon$ to a prefix of the string, the symbol **B** will be replaced by the two fresh symbols **W** (for “wait”) and $R_{i,j}$ where i represents the i^{th} rule and j means that ℓ_i has been split after j symbols (i.e. $|\ell_A| = j$). The fresh symbol **L** is used to signal that the suffix has been rewritten by rule $\ell_A \rightarrow r$. Finally, we use a copy of the

alphabet, to ensure completeness of the transformation: for an alphabet Σ , we denote by $\bar{\Sigma}$ a fresh copy of Σ , i.e. $\bar{\Sigma} = \{\bar{a} \mid a \in \Sigma\}$. For a word $w \in \Sigma^*$ with $\bar{w} \in \bar{\Sigma}^*$ we denote the word w where every symbol a is replaced by \bar{a} . Analogously, for a word $w \in \bar{\Sigma}^*$ with $\underline{w} \in \Sigma$ we denote w where every symbol \bar{a} is replaced by the symbol a .

► **Definition 3** (The transformation split). Let $R = \{\ell_1 \rightarrow r_1, \dots, \ell_n \rightarrow r_n\}$ be an SRS over alphabet Σ . Let $\bar{\Sigma}$ be a fresh copy of Σ and let $\mathbf{B}, \mathbf{E}, \mathbf{W}, \mathbf{R}_{i,j}, \mathbf{L}$ be fresh symbols (fresh for $\Sigma \cup \bar{\Sigma}$). The SRS $\text{split}(R)$ over alphabet $\Sigma \cup \bar{\Sigma} \cup \{\mathbf{B}, \mathbf{E}, \mathbf{L}, \mathbf{W}\} \cup \bigcup_{i=1}^n \{\mathbf{R}_{i,j} \mid 1 \leq j < |\ell_i|\}$ consists of the following string rewrite rules:

$$\ell_i \rightarrow r_i \quad \text{for every rule } (\ell_i \rightarrow r_i) \in R \quad (\text{splitA})$$

$$\bar{a}\mathbf{L} \rightarrow \mathbf{L}a \quad \text{for all } a \in \Sigma \quad (\text{splitB})$$

$$\mathbf{W}\mathbf{L} \rightarrow \mathbf{B} \quad (\text{splitC})$$

and for every rule $(\ell_i \rightarrow r_i) \in R$, for all $1 \leq j < |\ell_i|$ and $\ell_A \ell_B = \ell_i$ with $|\ell_A| = j$:

$$\mathbf{B}\ell_B \rightarrow \mathbf{W}\mathbf{R}_{i,j} \quad (\text{splitD})$$

$$\mathbf{R}_{i,j}\ell_A \mathbf{E} \rightarrow \mathbf{L}r_i \mathbf{E} \quad (\text{splitE})$$

$$\mathbf{R}_{i,j}a \rightarrow \bar{a}\mathbf{R}_{i,j} \quad \text{for all } a \in \Sigma \quad (\text{splitF})$$

We describe the intended use of the rules and the extra symbols. The symbols \mathbf{B} and \mathbf{E} mark the start and the end of the string, i.e. for a cycle $[u]$ the SRS $\text{split}(R)$ rewrites $\mathbf{B}u\mathbf{E}$.

Let $[u] \circ_{\rightarrow R} [w]$. The rule (splitA) covers the case that also $u \rightarrow_R w$ holds. Now assume that for $w' \sim w$ we have $u \xrightarrow{\{\ell_B \rightarrow \varepsilon\}} v \xrightarrow{\{\ell_A \rightarrow r\}} w'$ (where $(\ell_A \ell_B \rightarrow r) \in R$). Rule (splitD) performs the prefix rewrite step and replaces \mathbf{B} by \mathbf{W} to ensure that no other such a rule can be applied. Additionally, the symbol $\mathbf{R}_{i,j}$ corresponding to the rule and its splitting is added to ensure that only the right suffix rewrite step is applicable. Rule (splitF) moves the symbol $\mathbf{R}_{i,j}$ to right and rule (splitE) performs the suffix rewrite step. Rules (splitB) and (splitC) are used to finish the simulation of the cycle rewrite step by using the symbol \mathbf{L} to restore the original alphabet and to finally replace $\mathbf{W}\mathbf{L}$ by \mathbf{B} .

► **Example 4.** For $R_1 = \{aa \rightarrow aba\}$ the transformed string rewrite system $\text{split}(R_1)$ is:

$$\begin{array}{llll} aa & \rightarrow aba & (\text{splitA}) & \bar{a}\mathbf{L} \rightarrow \mathbf{L}a & (\text{splitB}) & \bar{b}\mathbf{L} \rightarrow \mathbf{L}b & (\text{splitB}) \\ \mathbf{W}\mathbf{L} & \rightarrow \mathbf{B} & (\text{splitC}) & \mathbf{B}a & \rightarrow \mathbf{W}\mathbf{R}_{1,1} & (\text{splitD}) & \mathbf{R}_{1,1}a\mathbf{E} \rightarrow \mathbf{L}aba\mathbf{E} & (\text{splitE}) \\ \mathbf{R}_{1,1}a & \rightarrow \bar{a}\mathbf{R}_{1,1} & (\text{splitF}) & \mathbf{R}_{1,1}b & \rightarrow \bar{b}\mathbf{R}_{1,1} & (\text{splitF}) & & \end{array}$$

For instance, the cycle rewrite step $[aba] \circ_{\rightarrow R_1} [baba]$ is simulated in the transformed system by $\mathbf{B}aba\mathbf{E} \rightarrow \mathbf{W}\mathbf{R}_{1,1}ba\mathbf{E} \rightarrow \mathbf{W}\bar{b}\mathbf{R}_{1,1}a\mathbf{E} \rightarrow \mathbf{W}\bar{b}\mathbf{L}aba\mathbf{E} \rightarrow \mathbf{W}\mathbf{L}baba\mathbf{E} \rightarrow \mathbf{B}baba\mathbf{E}$. As a further example, for $R_2 = \{abc \rightarrow cbacba, aa \rightarrow a\}$ the transformed SRS $\text{split}(R_2)$ is:

$$\begin{array}{llll} abc & \rightarrow cbacba & (\text{splitA}) & aa & \rightarrow a & (\text{splitA}) & \mathbf{W}\mathbf{L} & \rightarrow \mathbf{B} & (\text{splitC}) \\ \bar{a}\mathbf{L} & \rightarrow \mathbf{L}a & (\text{splitB}) & \bar{b}\mathbf{L} & \rightarrow \mathbf{L}b & (\text{splitB}) & \bar{c}\mathbf{L} & \rightarrow \mathbf{L}c & (\text{splitB}) \\ \mathbf{B}bc & \rightarrow \mathbf{W}\mathbf{R}_{1,1} & (\text{splitD}) & \mathbf{B}c & \rightarrow \mathbf{W}\mathbf{R}_{1,2} & (\text{splitD}) & \mathbf{B}a & \rightarrow \mathbf{W}\mathbf{R}_{2,1} & (\text{splitD}) \\ \mathbf{R}_{1,1}a\mathbf{E} & \rightarrow \mathbf{L}cbacba\mathbf{E} & (\text{splitE}) & \mathbf{R}_{1,2}ab\mathbf{E} & \rightarrow \mathbf{L}cbacba\mathbf{E} & (\text{splitE}) & \mathbf{R}_{2,1}a\mathbf{E} & \rightarrow \mathbf{L}a\mathbf{E} & (\text{splitE}) \\ \mathbf{R}_{1,1}a & \rightarrow \bar{a}\mathbf{R}_{1,1} & (\text{splitF}) & \mathbf{R}_{1,2}a & \rightarrow \bar{a}\mathbf{R}_{1,2} & (\text{splitF}) & \mathbf{R}_{2,1}a & \rightarrow \bar{a}\mathbf{R}_{2,1} & (\text{splitF}) \\ \mathbf{R}_{1,1}b & \rightarrow \bar{b}\mathbf{R}_{1,1} & (\text{splitF}) & \mathbf{R}_{1,2}b & \rightarrow \bar{b}\mathbf{R}_{1,2} & (\text{splitF}) & \mathbf{R}_{2,1}b & \rightarrow \bar{b}\mathbf{R}_{2,1} & (\text{splitF}) \\ \mathbf{R}_{1,1}c & \rightarrow \bar{c}\mathbf{R}_{1,1} & (\text{splitF}) & \mathbf{R}_{1,2}c & \rightarrow \bar{c}\mathbf{R}_{1,2} & (\text{splitF}) & \mathbf{R}_{2,1}c & \rightarrow \bar{c}\mathbf{R}_{2,1} & (\text{splitF}) \end{array}$$

Termination of $\text{split}(R_1)$ and $\text{split}(R_2)$ can be proved by AProVE and T₁T₂.

► **Proposition 5** (Soundness of split). *If $\rightarrow_{\text{split}(R)}$ is terminating then $\circ_{\rightarrow R}$ is terminating.*

Proof. By construction of $\text{split}(R)$, it holds that if $[u] \circ_{\rightarrow R} [v]$, then $\mathbf{B}u'\mathbf{E} \xrightarrow{+}_{\text{split}(R)} \mathbf{B}v'\mathbf{E}$ with $u \sim u'$ and $v \sim v'$. Thus for every infinite sequence $[w_1] \circ_{\rightarrow R} [w_2] \circ_{\rightarrow R} \dots$ there exists an infinite sequence $\mathbf{B} w'_1 \mathbf{E} \xrightarrow{\text{split}(R)} \mathbf{B} w'_2 \mathbf{E} \xrightarrow{\text{split}(R)} \dots$ with $w_i \sim w'_i$ for all i . ◀

3.1.1 Completeness of Split

We use type introduction [17] and use the sorts A , \bar{A} , C , and T , and type the symbols used by $\text{split}(R)$ (seen as unary function symbols, and seen as a term rewrite system) as follows:

$$\begin{array}{lll} \text{L} :: A \rightarrow \bar{A} & \text{W} :: \bar{A} \rightarrow T & a :: A \rightarrow A \text{ for all } a \in \Sigma \\ \text{B} :: A \rightarrow T & \text{E} :: C \rightarrow A & \bar{a} :: \bar{A} \rightarrow \bar{A} \text{ for all } \bar{a} \in \bar{\Sigma} \\ & & \text{R}_{i,j} :: A \rightarrow \bar{A} \text{ for all } \text{R}_{i,j} \end{array}$$

We also add a constant c of sort C to the alphabet, s.t. ground terms for any sort exist. First one can verify that all rewrite rules of $\text{split}(R)$ are well-typed: (splitA) rewrites terms of sort A , (splitB), (splitE), and (splitF) rewrite terms of sort \bar{A} , and (splitC) and (splitD) rewrite terms of sort T . Since there are no collapsing and duplicating rules, type introduction can be used (see [17]), i.e. (string) termination of the typed system is equivalent to (string) termination of the untyped system.

We consider ground terms of the sorts A , \bar{A} , C , and T . For simplicity we use the representation as strings (instead of terms), and we write E instead of Ec . First we show that our analysis of non-termination can be restricted to terms of sort T :

► **Lemma 6.** *If a term w of sort S with $S \in \{A, \bar{A}, C\}$ admits an infinite reduction w.r.t. $\text{split}(R)$, then there exists a term w' of sort T , which admits an infinite reduction.*

Proof. The only term of sort C is the constant c which is a normal form. If a term w of sort A admits an infinite reduction w.r.t. $\rightarrow_{\text{split}(R)}$, then also the term $\text{B}w$ of type T admits an infinite reduction w.r.t. $\rightarrow_{\text{split}(R)}$. If a term w of sort \bar{A} admits an infinite reduction w.r.t. $\rightarrow_{\text{split}(R)}$, then also the term $\text{W}w$ of type T admits an infinite reduction w.r.t. $\rightarrow_{\text{split}(R)}$. ◀

Inspecting the typing of the symbols shows:

► **Lemma 7.** *Any term of sort T is of one of the following forms:*

- $\text{B}u\text{E}$ where $u \in \Sigma^*$
- $\text{W}w\text{L}u\text{E}$ where $w \in \bar{\Sigma}^*$ and $u \in \Sigma^*$
- $\text{W}w\text{R}_{i,j}u\text{E}$ where $w \in \bar{\Sigma}^*$ and $u \in \Sigma^*$

We define a mapping from terms of sort T into strings over Σ as follows:

► **Definition 8.** For a term $w :: T$, the string $\Phi(w) \in \Sigma^*$ is defined according to the cases of Lemma 7:

$$\begin{array}{ll} \Phi(\text{B}u\text{E}) & := u \\ \Phi(\text{W}w\text{L}u\text{E}) & := \underline{w}u \\ \Phi(\text{W}w\text{R}_{i,j}u\text{E}) & := \ell_B \underline{w}u \quad \text{if } (\text{B}\ell_B \rightarrow \text{W}\text{R}_{i,j}) \in \text{split}(R) \end{array}$$

► **Lemma 9.** *Let w be of sort T and $w \rightarrow_{\text{split}(R)} w'$. Then $\Phi(w) \circ_{\rightarrow_R}^* \Phi(w')$.*

Proof. We inspect the cases of Lemma 7 for w :

- If $w = \text{B}u\text{E}$ where $u \in \Sigma^*$, then the step $w \rightarrow_{\text{split}(R)} w'$ can use a rule of type (splitA) or (splitD). If rule (splitA) is applied, then $\Phi(w) \rightarrow_R \Phi(w')$ and thus $\Phi(w) \circ_{\rightarrow_R} \Phi(w')$. If rule (splitD) is applied, then $w = \text{B}\ell_2 u' \rightarrow_{\text{split}(R)} \text{W}\text{R}_{i,j} u' = w'$ and $\Phi(w) = \ell_2 u' = \Phi(w')$.
- If $w = \text{W}v\text{L}u\text{E}$ where $v \in \bar{\Sigma}^*$ and $u \in \Sigma^*$, then the step $w \rightarrow_{\text{split}(R)} w'$ can use rules of type (splitA), (splitB), or (splitC). If rule (splitA) is used, then $\Phi(w) \rightarrow_R \Phi(w')$ and thus $\Phi(w) \circ_{\rightarrow_R} \Phi(w')$. If rule (splitB) or (splitC) is used, then $\Phi(w) = \Phi(w')$.

- If $w = WvR_{i,j}uE$ where $v \in \bar{\Sigma}^*$ and $u \in \Sigma^*$, then the step $w \rightarrow_{\text{split}(R)} w'$ can use a rule of type (splitA), (splitE), or (splitF). If rule (splitA) is used, then $\Phi(w) \rightarrow_R \Phi(w')$ and thus $\Phi(w) \circ \rightarrow_R \Phi(w')$. If rule (splitF) is used, then $\Phi(w) = \Phi(w')$. If rule (splitE) is used, then $w = WvR_{i,j}\ell_A E$ and $w' = WvLr_i E$ and $\Phi(w) = \ell_B \underline{v} \ell_A \sim \ell_A \ell_B \underline{v} \rightarrow_R r_i \underline{v} \sim \underline{v} r_i = \Phi(w')$ and thus $\Phi(w) \circ \rightarrow_R \Phi(w')$. ◀

► **Theorem 10** (Soundness and completeness of split). *The transformation split is sound and complete, i.e. $\rightarrow_{\text{split}(R)}$ is terminating if, and only if $\circ \rightarrow_R$ is terminating.*

Proof. Soundness is proved in Proposition 5. It remains to show completeness. W.l.o.g. we assume that \rightarrow_R is terminating, since otherwise $\circ \rightarrow_R$ is obviously non-terminating. Type introduction and Lemma 6 show that it is sufficient to construct a non-terminating cycle rewrite sequence for any term w of sort T where w has an infinite $\rightarrow_{\text{split}(R)}$ -reduction. For every infinite reduction $w \rightarrow_{\text{split}(R)} w_1 \rightarrow_{\text{split}(R)} w_2 \cdots$ we use Lemma 9 to construct a cycle rewrite sequence $\Phi(w) \circ \rightarrow_R^* \Phi(w_1) \circ \rightarrow_R^* \Phi(w_2) \cdots$. It remains to show that the constructed sequence is infinite: one can observe that the infinite sequence must have infinitely many applications of rule (splitE) (which is translated by $\Phi(\cdot)$ into exactly one $\circ \rightarrow_R$ -step), since every sequence of $\frac{(\text{splitA}) \vee (\text{splitB}) \vee (\text{splitC}) \vee (\text{splitD}) \vee (\text{splitF})}{\rightarrow}$ -steps is terminating (since we assumed that \rightarrow_R is terminating). ◀

3.2 Alternative Transformations

We first present the general ideas of the transformations rotate and shift before giving their definitions. We write \curvearrowright for the relation which moves the first element of a string to the end, i.e. $au \curvearrowright ua$ for every $a \in \Sigma$ and $u \in \Sigma^*$. We write $u \curvearrowright^{|\cdot|} v$ if, and only if $u \curvearrowright^{<|u|} v$ holds. Clearly, $u \sim v$ holds if and only if $u \curvearrowright^{<|u|} v$ holds. For a string rewrite system R , we define $\text{len}(R)$ as the size of the largest left-hand side of the rules in R , i.e. $\text{len}(R) = \max_{(\ell_i \rightarrow r_i) \in R} |\ell_i|$.

Using these notations we present two approaches to simulate cycle rewriting by string rewriting. One approach is to shift at most $\text{len}(R) - 1$ symbols from the left end to the right end and then to apply a string rewrite step (this is the relation $\curvearrowright^{<\text{len}(R)} \cdot \rightarrow_R$). Another approach is to first rotate the string and then to apply a prefix rewrite step (this is the relation $\curvearrowright^{|\cdot|} \cdot \hookrightarrow_R$).

► **Example 11.** As in Example 2, let $R = \{abc \rightarrow bbbb\}$ and $[bcdda] \circ \rightarrow_R [bddbb]$.

For the first approach (“shift”) we shift symbols from the left end to the right end of the string until abc becomes a substring. Then we apply a string rewrite step, i.e. $bcdda \curvearrowright^{<\text{len}(R)} \cdot \rightarrow_R ddbbbb$, since $bcdda \curvearrowright cddab \curvearrowright ddabc \rightarrow_R ddbbbb$.

For the second approach (“rotate”), we rotate the string (by iteratively shifting symbols) until abc becomes a prefix. Then we apply a prefix rewrite step, i.e. $bcdda \curvearrowright^{|\cdot|} \cdot \hookrightarrow_R bbbbdd$, since $bcdda \curvearrowright cddab \curvearrowright ddabc \curvearrowright dabcd \curvearrowright abcdd \hookrightarrow_R bbbbdd$.

It is quite easy to verify that the following proposition holds:

► **Proposition 12.** *Let R be an SRS. If $\circ \rightarrow_R$ is non-terminating, then*

1. $\curvearrowright^{<\text{len}(R)} \cdot \rightarrow_R$ admits an infinite reduction, and
2. $\curvearrowright^{|\cdot|} \cdot \hookrightarrow_R$ admits an infinite reduction.

For an SRS R the SRS $\text{shift}(R)$ encodes the relation $\curvearrowright^{<\text{len}(R)} \cdot \rightarrow_R$ and the SRS $\text{rotate}(R)$ encodes the relation $\curvearrowright^{|\cdot|} \cdot \hookrightarrow_R$ where extra symbols are used to separate the steps, and copies of the alphabet underlying R are used to ensure completeness of the transformations. We provide the definitions and some explanations of the transformations shift and rotate,

but for space reasons we exclude their proofs of soundness and completeness. They can be found in the appendix. For the remainder of the section, we fix an SRS R over alphabet $\Sigma_A = \{a_1, \dots, a_n\}$. Let us write $\Sigma_B, \Sigma_C, \Sigma_D, \Sigma_E$ for fresh copies of the alphabet Σ_A . We use the following notation to switch between the alphabets: for $X, Y \in \{A, B, C, D, E\}$ and $w \in \Sigma_X$ we write $\langle w \rangle_Y$ to denote the copy of w in the alphabet Y where every symbol is translated from alphabet X to alphabet Y .

► **Definition 13** (The transformation shift). Let R be an SRS over alphabet Σ_A and let $N = \max(0, \text{len}(R) - 1)$. The SRS $\text{shift}(R)$ over the alphabet $\Sigma_A \cup \Sigma_B \cup \Sigma_C \cup \{B, E, W, V, M, L, R, D\}$ (where B, E, W, V, M, L, R, D are fresh for $\Sigma_A \cup \Sigma_B \cup \Sigma_C$) consists of the following rules:

$$\begin{array}{ll}
B \rightarrow WM^N V & \text{(shiftA)} \\
M \rightarrow \varepsilon & \text{(shiftB)} \\
MVa \rightarrow V \langle a \rangle_B & \text{for all } a \in \Sigma_A \quad \text{(shiftC)} \\
ba \rightarrow ab & \text{for all } a \in \Sigma_A \text{ and all } b \in \Sigma_B \quad \text{(shiftD)} \\
bE \rightarrow \langle b \rangle_A E & \text{for all } b \in \Sigma_B \quad \text{(shiftE)} \\
WV \rightarrow RL & \text{(shiftF)} \\
La \rightarrow \langle a \rangle_C L & \text{for all } a \in \Sigma_A \quad \text{(shiftG)} \\
L\ell \rightarrow D r & \text{for all } (\ell \rightarrow r) \in R \quad \text{(shiftH)} \\
cD \rightarrow D \langle c \rangle_A & \text{for all } c \in \Sigma_C \quad \text{(shiftI)} \\
RD \rightarrow B & \text{(shiftJ)}
\end{array}$$

The rules (shiftA) - (shiftE) encode the relation $\curvearrowright^{<\text{len}(R)}$, i.e. for $uv \in \Sigma_A^*$ with $|u| < \text{len}(R)$, the string $BuvE$ is rewritten into $WVvuE$ by these five rules. The sequence of symbols M generated by rule (shiftA) denotes the potential of moving at most $\text{len}(R) - 1$ symbols. The rules (shiftB) and (shiftC) either remove one from the potential or start the moving of one symbol. The rule (shiftD) performs the movement of a single symbol until it reaches the end of the string and rule (shiftE) finishes the movement.

The remaining rewrite rules perform a single string rewrite step, i.e. for a rule $(\ell \rightarrow r) \in R$ the string $WVw_1 \ell w_2 E$ is rewritten to $Bw_1 r w_2 E$ by rules (shiftF) - (shiftJ).

This shows that if $u \curvearrowright^{<\text{len}(R)} v \rightarrow_R w$, then $BuE \xrightarrow{+}_{\text{shift}(R)} BwE$ and thus by Proposition 12 soundness of the transformation shift holds.

► **Definition 14** (The transformation rotate). Let R be an SRS over alphabet Σ_A . The SRS $\text{rotate}(R)$ over the alphabet $\Sigma_A \cup \Sigma_B \cup \Sigma_C \cup \Sigma_D \cup \Sigma_E \cup \{\text{Begin, End, Rewrite, Goright, Guess, Rotate, Cut, Moveleft, Wait, Finish, Finish2}\}$ (where $\text{Begin, End, Rewrite, Goright, Guess, Rotate, Cut, Moveleft, Wait, Finish}$, and Finish2 are fresh for $\Sigma_A \cup \Sigma_B \cup \Sigma_C \cup \Sigma_D \cup \Sigma_E$) is:

$$\begin{array}{ll}
\text{Begin End} \rightarrow \text{Rewrite End} & \text{(rotateA)} \\
\text{Begin } a \rightarrow \text{Rotate Cut } \langle a \rangle_D \text{ Guess} & \text{for all } a \in \Sigma_A \quad \text{(rotateB)} \\
\text{Guess } a \rightarrow \langle a \rangle_D \text{ Guess} & \text{for all } a \in \Sigma_A \quad \text{(rotateC)} \\
\text{Guess } a \rightarrow \text{Moveleft } \langle a \rangle_C \text{ Wait} & \text{for all } a \in \Sigma_A \quad \text{(rotateD)} \\
\text{Guess End} \rightarrow \text{Finish End} & \text{(rotateE)} \\
d \text{ Moveleft } c \rightarrow \text{Moveleft } c \langle d \rangle_B & \text{for all } c \in \Sigma_c \text{ and all } d \in \Sigma_D \quad \text{(rotateF)} \\
\text{Cut Moveleft } c \rightarrow \langle c \rangle_E \text{ Cut Goright} & \text{for all } c \in \Sigma_C \quad \text{(rotateG)} \\
\text{Goright } b \rightarrow \langle b \rangle_D \text{ Goright} & \text{for all } b \in \Sigma_B \quad \text{(rotateH)} \\
\text{Goright Wait } a \rightarrow \text{Moveleft } \langle a \rangle_C \text{ Wait} & \text{for all } a \in \Sigma_A \quad \text{(rotateI)} \\
\text{Goright Wait End} \rightarrow \text{Finish End} & \text{(rotateJ)} \\
d \text{ Finish} \rightarrow \text{Finish } \langle d \rangle_A & \text{for all } d \in \Sigma_D \quad \text{(rotateK)}
\end{array}$$

CutFinish \rightarrow Finish2		(rotateL)
e Finish2 \rightarrow Finish2(e) _A	for all $e \in \Sigma_E$	(rotateM)
RotateFinish2 \rightarrow Rewrite		(rotateN)
Rewrite ℓ \rightarrow Begin r	for all $(\ell \rightarrow r) \in R$	(rotateO)

We describe the intended use of the rewrite rules, where we ignore the copies of the alphabet in our explanations. The goal is that for any string $w \in \Sigma_A^*$, the string **Begin** w **End** is rewritten to **Begin** u **End**, where $w \curvearrowright^{| \cdot |} \hookrightarrow_R u$. The prefix rewrite step is performed by the last rule (rotateO). All other rules perform the rotation $\curvearrowright^{| \cdot |}$ s.t. **Begin** w **End** is rewritten into **Rewrite** v **End** where $w \sim v$. Instead of moving symbols from the front to the end (as \curvearrowright does), the rules move a suffix of the string in front of the string (which has the same effect).

The first rewrite rule (rotateA) covers the case that w is empty. If $w = a_1 \dots a_n$, then first choose a position to cut the string into two parts $w_1 w_2$. The symbol **Guess** is used for the non-deterministic selection of the position. Rule (rotateB) starts the rotate phase and the guessing, rule (rotateC) shifts the **Guess**-marker and rule (rotateD) stops the guessing. Rule (rotateE) covers the case that $w_2 = \varepsilon$ and no rotation will be performed. After stopping the guessing, every symbol of w_2 is moved in front of w_1 , resulting in $w_2 w_1$. A typical situation is $a_{k+1} \dots a_m a_1 \dots a_k a_{m+1} \dots a_n$ and now the symbol a_{m+1} must be moved in between a_m and a_1 . To keep track of the position of a_1 , the symbol **Cut** (inserted in front of a_1) marks the original beginning, and to keep track of the position of a_k , the symbol **Wait** (inserted after a_k) marks this position. The symbol **Movelf** guards the movement of a_{m+1} (by rule (rotateF)). When arriving at the right place (rule (rotateG)), the symbol **Goright** is used to walk along the string (rule (rotateH)) to find the next symbol which has to be moved (rule (rotateI)). If all symbols are moved, rule (rotateJ) is applied to start the clean-up phase. There the symbols **Finish** and **Finish2** are used to remove the markers and to replace the copied symbols of the alphabet with the original ones (rules (rotateK) – (rotateN)).

The construction of $\text{rotate}(R)$ shows that **Begin** u **End** $\xrightarrow{*}_{\text{rotate}(R)}$ **Begin** w **End** whenever $u \curvearrowright^{| \cdot |} v \hookrightarrow_R w$. Thus Proposition 12 implies soundness of **rotate**.

In Appendix A we prove soundness and completeness of both transformations **shift** and **rotate** and thus the following theorem holds:

► **Theorem 15** (see). *The transformations **shift** and **rotate** are sound and complete.*

4 Trace Decreasing Matrix Interpretations

In this section we present a variant of matrix interpretations suitable for proving cycle termination. The basics of matrix interpretations for string and term rewriting were presented in [9, 10, 5, 11]. The special case of tropical and arctic matrix interpretations for cycle rewriting was presented in [18], in the setting of *type graphs*. This section extends matrix interpretations for cycle rewriting along the lines suggested by Johannes Waldmann.

Fix a dimension $d > 0$. Define M_d to be the set of $d \times d$ matrices A over \mathbf{N} for which $A_{11} > 0$. On M_d we define the relations $>$ and \geq by

$$A > B \iff A_{11} > B_{11} \wedge \forall i, j : A_{ij} \geq B_{ij}, \quad A \geq B \iff \forall i, j : A_{ij} \geq B_{ij}.$$

Write \times for matrix multiplication. Note that $(A \times B) \in M_d$ whenever $A, B \in M_d$. The following lemma is easily checked.

► **Lemma 16.** *Let $A, B, C \in M_d$.*

- *If $A > B$ then $A \times C > B \times C$ and $C \times A > C \times B$,*

- If $A \geq B$ then $A \times C \geq B \times C$ and $C \times A \geq C \times B$.

A *matrix interpretation* $\langle \cdot \rangle$ for a signature Σ is defined to be a mapping from Σ to M_d . It is extended to $\langle \cdot \rangle : \Sigma^* \rightarrow M_d$ by defining inductively $\langle \varepsilon \rangle = I$ and $\langle ua \rangle = \langle u \rangle \times \langle a \rangle$ for all $u \in \Sigma^*$, $a \in \Sigma$, where I is the identity matrix.

► **Theorem 17.** *Let $R' \subseteq R$ be SRSs over Σ and let $\langle \cdot \rangle : \Sigma \rightarrow M_d$ such that*

- $\circ \rightarrow_{R'}$ is terminating,
- $\langle \ell \rangle \geq \langle r \rangle$ for all $(\ell \rightarrow r) \in R'$, and
- $\langle \ell \rangle > \langle r \rangle$ for all $(\ell \rightarrow r) \in R \setminus R'$.

Then $\circ \rightarrow_R$ is terminating.

Proof. For a square matrix A of dimension d , its *trace* $\text{tr}(A)$ is defined to be $\sum_{i=1}^d A_{ii}$: the sum of its diagonal. It is well known and easy to check that $\text{tr}(A \times B) = \text{tr}(B \times A)$ for all A, B . As a consequence we obtain $\text{tr}(\langle u \rangle) = \text{tr}(\langle v \rangle)$ for u, v satisfying $u \sim v$. Since $A > B$ implies $\text{tr}(A) > \text{tr}(B)$ and $A \geq B$ implies $\text{tr}(A) \geq \text{tr}(B)$, from Lemma 16 we obtain $\text{tr}(\langle u \rangle) \geq \text{tr}(\langle v \rangle)$ if $u \rightarrow_{R'} v$, and $\text{tr}(\langle u \rangle) > \text{tr}(\langle v \rangle)$ if $u \rightarrow_{R \setminus R'} v$. Combining these observations yields $\text{tr}(\langle u \rangle) \geq \text{tr}(\langle v \rangle)$ if $[u] \circ \rightarrow_{R'} [v]$, and $\text{tr}(\langle u \rangle) > \text{tr}(\langle v \rangle)$ if $[u] \circ \rightarrow_{R \setminus R'} [v]$.

Assume an infinite $\circ \rightarrow_R$ reduction exists: $[u_1] \circ \rightarrow_R [u_2] \circ \rightarrow_R [u_3] \circ \rightarrow_R [u_4] \circ \rightarrow_R \dots$. Since $\circ \rightarrow_{R'}$ is terminating, it contains infinitely many steps $[u_i] \circ \rightarrow_{R \setminus R'} [u_{i+1}]$, all giving rise to $\text{tr}(\langle u_i \rangle) > \text{tr}(\langle u_{i+1} \rangle)$, while all other steps give rise to $\text{tr}(\langle u_i \rangle) \geq \text{tr}(\langle u_{i+1} \rangle)$. As $\text{tr}(\langle \cdot \rangle)$ always yields a natural number, this yields an infinite descending sequence of natural numbers, contradiction. ◀

Although this proof is not very hard, it is quite subtle: it is essential to first use the full order on matrices and disallow A_{11} to be 0 in order to obtain Lemma 16, and next apply the trace to get the same interpretations for u and v if $u \sim v$. It is not essential to apply this approach to natural numbers with usual addition and multiplication: other well-founded semirings can be used as well. Using the semiring $\mathbf{N} \cup \{\infty\}$ with minimum as semiring addition and $+$ as semiring multiplication yields *tropical matrix interpretations*. Using the semiring $\mathbf{N} \cup \{-\infty\}$ with maximum as semiring addition and $+$ as semiring multiplication yields *arctic matrix interpretations*. For general theory on matrix interpretations for term rewriting we refer to [10, 5]. Validity of tropical and arctic matrix interpretations for cycle rewriting has been proved in [18], in the setting of *type graphs*.

The original versions of matrix interpretations in [9, 5] fail for proving cycle termination since they succeed in proving termination of $aa \rightarrow bc$, $bb \rightarrow ac$, $cc \rightarrow ab$, for which cycle termination does not hold due to $[caa] \circ \rightarrow [baa] \circ \rightarrow [bbc] \circ \rightarrow [acc]$. The main difference is that in our setting the interpretation of symbols is multiplication by a matrix, while in [9, 5] it combines such a matrix multiplication by adding a vector.

The search for an application of Theorem 17 has been implemented in our tool `torpacyc`, extending the version presented in [18]. It is done by transforming the requirements to SMT format and calling the external SMT solver `Yices` [3, 16]. As an example consider the system from the introduction:

$$0P \rightarrow 1P, 1P \rightarrow cP, 0c \rightarrow 10, 1c \rightarrow c0, P0 \rightarrow P100.$$

First by finding a tropical matrix interpretation, the last rule is removed by `torpacyc`. Next the following matrices are found:

$$\langle P \rangle = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \langle 0 \rangle = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \langle 1 \rangle = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \langle c \rangle = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

For these interpretations we obtain $\langle 0P \rangle > \langle 1P \rangle$, $\langle 1P \rangle > \langle cP \rangle$, $\langle 0c \rangle \geq \langle 10 \rangle$ and $\langle 1c \rangle \geq \langle c0 \rangle$, hence by Theorem 17 it suffices to prove cycle termination of $0c \rightarrow 10$, $1c \rightarrow c0$, for which `torpacyc` finds a simple counting argument.

4.1 Limitations of the Method

In this section we give an example where the matrix approach fails and the transformational approach succeeds.

The method for proving cycle termination induced by Theorem 17 has similar limitations as the method of matrix interpretations in [10] for string termination: Since the entries of a product of n matrices are bounded by an exponential function in n , the method cannot prove cycle termination of systems which allow reduction sequences where every rewrite rule is applied more often than exponentially often.

► **Example 18.** The rewrite system $R_1 := \{ab \rightarrow bca, cb \rightarrow bbc\}$ allows for string derivations of a length which is a tower of exponentials (see [10]), i.e. the string $a^k b^k$ has such a long derivation, since the derivation $ab^n \xrightarrow{*}_{R_1} b^{2^n-1} c^n a$ exists and this can be iterated for every a in a^k . Moreover, the number of applications of the first and of the second rule of R_1 is a tower of exponentials. This shows that the matrix interpretations in [10] are unable to prove string termination of R_1 . The system $\phi(R_1) := \{RE \rightarrow LE, aL \rightarrow La', bL \rightarrow Lb', cL \rightarrow Lc', Ra' \rightarrow aR, Rb' \rightarrow bR, Rc' \rightarrow cR, abL \rightarrow bcaR, cbL \rightarrow bbcR\}$ uses the transformation ϕ from [18] and transforms the string rewrite system R_1 into a cycle rewrite system s.t. R_1 is string terminating iff $\phi(R_1)$ is cycle terminating. One can verify that $[ab^n LE] \circ \xrightarrow{*}_{\phi(R_1)} [b^{2^n-1} c^n aLE]$ which can also be iterated s.t. $[a^k b^k LE]$ has a cycle rewriting sequence whose length is a tower of k exponentials. Inspecting all nine rules of $\phi(R_1)$, the number of applications of any of the rules in this rewrite sequence is also a tower of k exponentials and thus it is impossible to prove cycle termination of $\phi(R_1)$ using Theorem 17. Consequently, our tool `torpacyc` does not find a termination proof for $\phi(R_1)$.

► **Remark 19.** As expected our tool `torpacyc` does not find a termination proof for $\phi(R_1)$ from Example 18. On the other hand, with our transformational approach a cycle termination can be proved: AProVE proves strings termination of `split($\phi(R_1)$)`.

A further question is whether matrix interpretations are limited to cycle rewrite systems with exponential derivation lengths only. The following example shows that this is not true:

► **Example 20.** The SRS $R_2 := \{ab \rightarrow baa, cb \rightarrow bbc\}$ (see [10]) has derivations of doubly exponential length (since $ac^k b \xrightarrow{*}_{R_2} b^{2^k} a^{2^{2^k}} c^k$ and any rewrite step adds one symbol), but its string termination can be proved by relative termination and matrix interpretations by first removing the rule $cb \rightarrow bbc$ and then removing the other rule. This is possible, since the second rule is applied only exponentially often. For cycle rewriting the encoding ϕ from [18] is $\phi(R_2) = \{RE \rightarrow LE, aL \rightarrow La', bL \rightarrow Lb', cL \rightarrow Lc', Ra' \rightarrow aR, Rb' \rightarrow bR, Rc' \rightarrow cR, abL \rightarrow baaR, cbL \rightarrow bbcR\}$ and $\phi(R_2)$ is cycle terminating iff R_2 is string terminating. The system $\phi(R_2)$ also has doubly exponential cycle derivations, e.g. $[ac^k bLE] \circ \xrightarrow{*}_{\phi(R_2)} [b^{2^k} a^{2^{2^k}} c^k LE]$. However, `torpacyc` proves cycle termination of $\phi(R_2)$ by first removing the last rule using the matrix interpretation

$$\begin{aligned} \langle R \rangle &= \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \langle E \rangle = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \langle L \rangle = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \langle a \rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \langle a' \rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \langle b \rangle &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \langle b' \rangle = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \langle c \rangle = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \langle c' \rangle = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}. \end{aligned}$$

Thereafter the remaining rules (which now only have derivations of exponential length) are eliminated by matrix interpretations and counting arguments.

Note that the methods in [18] are not able to prove cycle termination of $\phi(R_2)$, since they can only remove rules which are applied polynomially often in any derivation.

Also the transformational approach successfully proves cycle termination of $\phi(R_2)$: T_1T_2 proves string termination of $\text{split}(\phi(R_2))$. Interestingly, we did not find a termination proof of $\text{split}(\phi(R_2))$ using AProVE.

5 Tools and Experimental Results

We implemented the search for matrix interpretations described in Section 4 in `torpacyc`. We also implemented a tool which transforms an SRS by `split`, `rotate`, or `shift`. We also automatized the proof of cycle termination by a command line tool which can either use `torpacyc` to prove cycle termination or by first performing one of the transformations, and then using one of the termination provers AProVE or T_1T_2 to prove string termination of the transformed problem. Also two kinds of combinations of both approaches are implemented:

- variant 1: first `torpacyc` tries to find a termination proof and if no proof is found, the (perhaps simplified) cycle rewrite system is transformed into a string rewrite system by `split` and then used as input for AProVE or T_1T_2 .
- variant 2: first string non-termination of the rewrite system is checked by AProVE or T_1T_2 . If a non-termination proof is found, then also the cycle rewrite system is non-terminating. Otherwise, the same procedure as in variant 1 is used.

The automatic transformation and the prover can also be used online via a web interface available via <http://www.ki.informatik.uni-frankfurt.de/research/cycsrs/> where also the tools and experimental data can be found.

5.1 Experiments

A problem for doing experiments is that no real appropriate test set is available. We played around with several examples like the ones in this paper, but we were also interested in larger scale experiments. To that end we chose two problem sets. The first set is the `SRS_Standard`-branch of the Termination Problem Data Base [14], which is a benchmark set for proving termination of string rewrite systems. The second set consists of 50 000 randomly generated cycle rewrite systems of size 12 over an alphabet of size 3. For all problems we tried to prove cycle termination using the following methods (all with a time limit of 60 seconds):

- `torpacyc`: We applied `torpacyc` to the problems, where version 2014 is described in [18] and version 2015 includes the matrix interpretations described in Section 4.
- `split, rotate, shift`: We transformed the problem by the transformation into a string termination problem and then applied the termination provers AProVE or T_1T_2 , respectively.
- combinations: We also tried the combination of the methods as described before (using AProVE and T_1T_2) where for variant 1, `torpacyc` as well as the second termination prover had a time limit of 30 seconds, and for variant 2, both – the non-termination check and `torpacyc` – had a time limit of 15 seconds, and the termination prover applied to the transformed problem (i.e. AProVE or T_1T_2) had a time limit of 30 seconds.

	torpacyc		split		rotate		shift		combination				string termination		
	2014	2015	AProVE	T ₁ T ₂	AProVE	T ₁ T ₂	AProVE	T ₁ T ₂	variant 1		variant 2		any	AProVE	T ₁ T ₂
									AProVE	T ₁ T ₂	AProVE	T ₁ T ₂			
yes	36	46	40	30	10	6	10	8	55	55	54	54	63	652	627
no	0	0	309	168	45	0	65	0	310	161	335	173	336	97	38
maybe	1289	1269	966	1117	1260	1309	1240	1307	950	1099	926	1088	916	566	650

■ **Table 1** Results of proving cycle termination of the problems in SRS_Standard of the TPDB

5.1.1 String Rewrite Systems of the Termination Problem Data Base

We tested all 1315 string rewrite problems of the SRS_Standard-branch of the Termination Problem Data Base [14]. Table 1 shows the summary of the performed tests, where in the column titled with “any” the output of all tools are combined (per problem). The last two columns show the results of proving string termination of the original problems in our test environment using AProVE and T₁T₂. Note that a non-termination proof of string rewriting also implies non-termination of cycle rewriting, which does not hold for termination proofs. The results show that having sound and complete transformations is advantageous (compared to having sound transformations only), since we were able to prove cycle non-termination for 336 problems by the transformational approach (and by using string non-termination). Note that torpacyc (in both version) has no technique to prove non-termination.

However, the results also show that most of the problems in the test set are too hard to be proved by the techniques (only 399 out of 1315 problems were shown to be terminating or non-terminating). This is not really surprising since the test set contains already ‘hard’ instances for proving string termination as shown by the results in the last two columns. Moreover, a substantial part of the problems is expected not to be cycle terminating, for instance by containing a renaming of $ab \rightarrow ba$. This is confirmed by the result of non-termination by split and AProVE for over 300 of the systems.

Comparing the three transformations, the transformation split leads to much better results than the other two transformations, which holds for termination and for non-termination proofs. Ignoring the non-termination results (since torpacyc does not check for non-termination), the following observations are made: the new version of torpacyc indeed improves the former version, the power of the transformation split together with AProVE compared to torpacyc 2015 is more or less equal, while other transformations and back-ends do not perform as well as these tools. The combination of techniques increases the power, not only since the different tools perform well on different problems, but also, since torpacyc passes its output (a perhaps simplified SRS) to the string termination prover.

5.1.2 Randomly Generated String Rewrite Systems

As a further test set which contains much simpler problems than the former test set, we generated 50 000 SRSs of size 12 over an alphabet of size 3, where only SRSs with at least one rule ($\ell \rightarrow r$) with $|r| \geq |\ell|$ are considered (to rule out obviously terminating problems), SRSs with rules ($\ell \rightarrow ulv$) are not considered (to rule out obviously non-terminating problems), only SRSs without collapsing rules are considered (since torpacyc 2014 cannot handle such problems), and no alpha-equivalent (i.e. renamed and reordered) SRSs are generated.

The summarized results of applying our methods to this problem set are shown in Table 2.

	combination												any
	torpacyc		split		rotate		shift		variant 1		variant 2		
	2014	2015	AProVE	$\top\top_2$	AProVE	$\top\top_2$	AProVE	$\top\top_2$	AProVE	$\top\top_2$	AProVE	$\top\top_2$	
yes	46981	46929	47017	47073	36967	36260	37053	37027	47071	46967	47064	47015	47124
no	0	0	2331	2201	184	0	771	0	2328	2011	2334	2174	2339
maybe	3019	3071	652	726	12849	13740	12176	12973	601	1022	602	811	537

■ **Table 2** Results of proving cycle termination of 50 000 randomly generated problems of size 12 over an alphabet of size 3

Most of the problems are cycle terminating and both versions of `torpacyc` as well as the transformation `split` together with AProVE or $\top\top_2$ can show termination of most of the problems. The problems seem to be too simple to separate the power of these successful approaches and their combinations. However, for the two transformations `rotate` and `shift`, the results show their lack in power, since no proof is found for roughly a quarter of all problems.

6 Conclusions

We developed new techniques to prove cycle termination. The main approach is to reduce the problem of cycle termination to the problem of string termination by applying a sound and complete transformation from cycle into string rewriting. We presented and analyzed three such transformations. Apart from that we provided a variant of matrix interpretations which improves the approach presented in [18]. Our implementations and the corresponding experimental results show that both techniques are useful in the sense that they apply for several examples for which the earlier techniques failed.

Together with the sound and complete transformation ϕ in the reverse direction from [18], the existence of a sound and complete transformation like `split` implies that the problems of cycle termination and string termination of SRSs are equivalent in a strong sense. For instance, it implies that they are in the same level of the arithmetic hierarchy, which is Π_2^0 -complete along the lines of [4]. Alternatively, Π_2^0 -completeness of cycle termination can be concluded from the sound and complete transformation ϕ combined with the observation that cycle termination is in Π_2^0 .

Acknowledgments

We thank Johannes Waldmann for fruitful remarks, in particular for his suggestions leading to Section 4 on trace decreasing matrix interpretations. We also thank the anonymous reviewers for there valuable comments.

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A Soundness and Completeness of Rotate and Shift

A.1 Proof of Proposition 12

The following lemma obviously holds:

- **Lemma 21.** *For any alphabet Σ and all $u, v \in \Sigma^*$: $u \sim v$ iff $u \curvearrowright^{<|u|} v$.
Moreover, for an SRS R over an alphabet Σ and $u, v \in \Sigma^*$:*
- *If $[u] \circ \rightarrow_R [v]$ then there exist $u', v' \in \Sigma^*$ s.t. $u \curvearrowright^* u' \hookrightarrow_R v' \curvearrowright^* v'$*
 - *If $[u] \circ \rightarrow_R [v]$ then there exist $u', v' \in \Sigma^*$ s.t. $u \curvearrowright^* u' \rightarrow_R v' \curvearrowright^* v'$.*

We can restrict the number of \curvearrowright -steps, before a rewrite or prefix-rewrite step:

- **Lemma 22.** *Let R be an SRS over alphabet Σ , and $u, v, w \in \Sigma^*$.*
1. *If $u \curvearrowright^k v \rightarrow_R w$ for some $k \geq \text{len}(R)$, then $u \curvearrowright^m v' \rightarrow_R v'' \curvearrowright^* w$ for some $v', v'' \in \Sigma^*$ and $m < \text{len}(R)$.*
 2. *If $u \curvearrowright^k v \hookrightarrow_R w$ for some $k \geq |u|$, then $u \curvearrowright^m v' \hookrightarrow_R v'' \curvearrowright^* w$ for some $v', v'' \in \Sigma^*$ and $m < |u|$.*

Proof of Proposition 12. The claim is:

Let R be an SRS. If $\circ \rightarrow_R$ is non-terminating, then

1. $\curvearrowright^{<\text{len}(R)}$. \rightarrow_R admits an infinite reduction, and
2. $\curvearrowright^{<| \cdot |}$. \hookrightarrow_R admits an infinite reduction.

Let $[w_1] \circ \rightarrow_R [w_2] \circ \rightarrow_R \dots$ be an infinite cycle rewrite sequence.

1. By Lemma 21 we have $w_1 \curvearrowright^* w'_1 \rightarrow_R w''_1 \curvearrowright^* w_2 \curvearrowright^* w'_2 \rightarrow_R w''_2 \curvearrowright^* \dots$. Lemma 22 shows that we can always move more than $\text{len}(R)$ \curvearrowright -steps to the right, and thus we derive an infinite sequence $w_1 \curvearrowright^{<\text{len}(R)} u_1 \rightarrow_R u'_1 \curvearrowright^{<\text{len}(R)} u_2 \rightarrow_R u'_2 \dots$ and thus w_1 admits an infinite reduction of the required form.
2. By Lemma 21 we have $w_1 \curvearrowright^* u_1 \hookrightarrow_R v_1 \curvearrowright^* w_2 \curvearrowright^* u_2 \hookrightarrow_R v_2 \curvearrowright^* \dots$, i.e. (by joining the \curvearrowright^* -steps) we get $v_0 = w_1 \curvearrowright^* u_1 \hookrightarrow_R v_1 \curvearrowright^* u_2 \hookrightarrow_R v_2 \curvearrowright^* \dots$. Lemma 22 shows that we can always move more than $|v_i| - 1$ \curvearrowright -steps to the right, and thus we derive an infinite sequence $v_0 \curvearrowright^{<|v_0|} u_1 \hookrightarrow_R v_1 \curvearrowright^{<|v_1|} u_2 \hookrightarrow_R v_3 \dots$ and thus v_0 admits an infinite reduction of the required form. ◀

A.2 Soundness and Completeness of Rotate

With $\text{rot}(\Sigma_A)$ we denote the string rewrite system consisting of the rules (rotateA) – (rotateN) from Definition 14 (excluding the rule (rotateO)).

- **Lemma 23.** *The SRS $\text{rot}(\Sigma_A)$ is terminating.*

Proof. Let σ be the following polynomial interpretation:

$$\begin{array}{lll}
 \sigma(\text{Rewrite}) = \lambda x.x & \sigma(\text{Goright}) = \lambda x.2x & \sigma(\text{Finish}) = \lambda x.2x + 2 \\
 \sigma(\text{Finish2}) = \lambda x.2x + 2 & \sigma(\text{End}) = \lambda x.x + 1 & \sigma(\text{Moveleft}) = \lambda x.2x + 1 \\
 \sigma(\text{Wait}) = \lambda x.3x & \sigma(\text{Rotate}) = \lambda x.x + 9 & \sigma(\text{Cut}) = \lambda x.2x \\
 \sigma(\text{Guess}) = \lambda x.6x + 1 & \sigma(\text{Begin}) = \lambda x.12x + 18 & \\
 \sigma(f) = \lambda x.4x + 1 & \text{for all } f \in (\Sigma_A \cup \Sigma_B \cup \Sigma_C \cup \Sigma_D \cup \Sigma_E) &
 \end{array}$$

We verify that for every rule $(\ell \rightarrow r) \in \text{rot}(\Sigma_A)$ the inequality $\sigma(\ell) > \sigma(r)$ holds:

- rule (rotateA): $\lambda x.12x + 30 > \lambda x.x + 1$,
- rule (rotateB): $\lambda x.48x + 30 > \lambda x.48x + 19$,
- rule (rotateC): $\lambda x.24x + 7 > \lambda x.24x + 5$,
- rule (rotateD): $\lambda x.24x + 7 > \lambda x.24x + 3$,
- rule (rotateE): $\lambda x.6x + 7 > \lambda x.2x + 4$,
- rule (rotateF): $\lambda x.32x + 13 > \lambda x.32x + 11$,
- rule (rotateG): $\lambda x.16x + 6 > \lambda x.16x + 1$,
- rule (rotateH): $\lambda x.8x + 2 > \lambda x.8x + 1$,
- rule (rotateI): $\lambda x.24x + 6 > \lambda x.24x + 3$,
- rule (rotateJ): $\lambda x.6x + 6 > \lambda x.2x + 4$,
- rule (rotateK): $\lambda x.8x + 4 > \lambda x.8x + 4$,
- rule (rotateL): $\lambda x.4x + 4 > \lambda x.2x + 2$,
- rule (rotateM): $\lambda x.8x + 4 > \lambda x.8x + 4$,
- rule (rotateN): $\lambda x.2x + 11 > \lambda x.x$.

Thus termination of $\text{rot}(\Sigma_A)$ has been proved. ◀

► **Proposition 24.** *If $u \curvearrowright^{| \cdot |} v$ then $\text{Begin } u \text{ End} \rightarrow_{\text{rot}(\Sigma_A)}^* \text{Rewrite } v \text{ End}$.*

Proof. If $u = \varepsilon$ then $v = \varepsilon$ and $\text{Begin End} \rightarrow_{\text{rot}(\Sigma_A)} \text{Rewrite End}$. If $u = a_1 \dots a_n$ for some $n \geq 1$ and $v = a_k \dots a_n a_1 \dots a_{k-1}$ then there are two cases: If $k = 1$ (i.e. $u \curvearrowright^0 v$) then the following rewrite sequence for $\text{Begin } u \text{ End}$ exists:

$$\begin{aligned} & \text{Begin } a_1 \dots a_n \text{ End} \rightarrow_{\text{rot}(\Sigma_A)} \text{RotateCut } \langle a_1 \rangle_D \text{ Guess } a_2 \dots a_n \text{ End} \\ & \rightarrow_{\text{rot}(\Sigma_A)}^* \text{RotateCut } \langle a_1 \rangle_D \dots \langle a_n \rangle_D \text{ GuessEnd} \rightarrow_{\text{rot}(\Sigma_A)}^* \text{RotateCut } \langle a_1 \rangle_D \dots \langle a_n \rangle_D \text{ GuessEnd} \\ & \rightarrow_{\text{rot}(\Sigma_A)} \text{RotateCut } \langle a_1 \rangle_D \dots \langle a_n \rangle_D \text{ FinishEnd} \rightarrow_{\text{rot}(\Sigma_A)}^* \text{Rewrite } a_1 \dots a_n \text{ End} \end{aligned}$$

If $1 < k \leq n$ then the following rewrite sequence for $\text{Begin } u \text{ End}$ exists:

$$\begin{aligned} & \text{Begin } a_1 \dots a_n \text{ End} \rightarrow_{\text{rot}(\Sigma_A)} \text{RotateCut } \langle a_1 \rangle_D \text{ Guess } a_2 \dots a_n \text{ End} \\ & \rightarrow_{\text{rot}(\Sigma_A)}^* \text{RotateCut } \langle a_1 \rangle_D \dots \langle a_{k-1} \rangle_D \text{ Guess } a_k \dots a_n \text{ End} \\ & \rightarrow_{\text{rot}(\Sigma_A)} \text{RotateCut } \langle a_1 \rangle_D \dots \langle a_{k-1} \rangle_D \text{ Moveleft } \langle a_k \rangle_C \text{ Wait } a_{k+1} \dots a_n \text{ End} \\ & \rightarrow_{\text{rot}(\Sigma_A)}^* \text{Rotate } \langle a_k \rangle_E \dots \langle a_n \rangle_E \text{ Cut } \langle a_1 \rangle_D \dots \langle a_{k-1} \rangle_D \text{ GorightWaitEnd} \\ & \rightarrow_{\text{rot}(\Sigma_A)} \text{Rotate } \langle a_k \rangle_E \dots \langle a_n \rangle_E \text{ Cut } \langle a_1 \rangle_D \dots \langle a_{k-1} \rangle_D \text{ FinishEnd} \\ & \rightarrow_{\text{rot}(\Sigma_A)}^* \text{Rewrite } a_k \dots a_n a_1 \dots a_{k-1} \text{ End} \end{aligned} \quad \blacktriangleleft$$

► **Proposition 25.** *If $u \curvearrowright^{| \cdot |} v \hookrightarrow_R w$ then $\text{Begin } u \text{ End} \rightarrow_{\text{rotate}(R)}^* \text{Begin } w \text{ End}$.*

Proof. This follows from Proposition 24 and the construction of $\text{rotate}(R)$. ◀

Thus by Proposition 12 we have

► **Theorem 26.** *The transformation rotate is sound.*

A.2.1 Completeness of Rotate

We write $\widehat{\Sigma}_A$ for the extension of alphabet Σ_A by the fresh symbols: Begin , End , Rewrite , Goright , Guess , Rotate , Cut , Moveleft , Wait , Finish , Finish2 and by four copies $\Sigma_B, \Sigma_C, \Sigma_D, \Sigma_E$ of the alphabet Σ_A .

For proving completeness of the transformation rotate we use type introduction using sorts $T_0, T_1, T_2, T_3, T_4, T_5, T_6$ and interpret the string rewrite system as a term rewrite system.

We add a constant c of sort T_0 (and thus ground terms of any sort exist) and use the following types for all symbols of $\widehat{\Sigma}_A$:

a	:: $T_1 \rightarrow T_1$	for all $a \in \Sigma_A$	End	:: $T_0 \rightarrow T_1$	Finish	:: $T_1 \rightarrow T_4$
b	:: $T_2 \rightarrow T_2$	for all $b \in \Sigma_B$	Wait	:: $T_1 \rightarrow T_2$	Rewrite	:: $T_1 \rightarrow T_6$
c	:: $T_2 \rightarrow T_3$	for all $c \in \Sigma_C$	Moveleft	:: $T_3 \rightarrow T_4$	Begin	:: $T_1 \rightarrow T_6$
d	:: $T_4 \rightarrow T_4$	for all $d \in \Sigma_D$	Goright	:: $T_2 \rightarrow T_4$	Rotate	:: $T_5 \rightarrow T_6$
e	:: $T_5 \rightarrow T_5$	for all $e \in \Sigma_E$	Cut	:: $T_4 \rightarrow T_5$	Finish2	:: $T_1 \rightarrow T_5$

One can verify that for any R the SRS $\text{rotate}(R)$ is well-typed: rules (rotateA), (rotateB), (rotateN), and (rotateO) rewrite terms of sort T_6 ; rules (rotateG), (rotateL), and (rotateM) rewrite terms of sort T_5 ; and all other rules rewrite terms of sort T_4 . Thus type introduction can be used, i.e. $\text{rotate}(R)$ is terminating if, and only if the typed system terminates. Again we use the representation as strings (instead of terms) and write **End** instead of **Endc**.

► **Lemma 27.** *Let $w \in \widehat{\Sigma}_A^*$ be well-typed, s.t. w admits an infinite reduction w.r.t. $\rightarrow_{\text{rotate}(R)}$. Then there exists a well-typed term $w' \in \widehat{\Sigma}_A^*$ of sort T_6 which admits an infinite reduction.*

Proof. If w is not of sort T_6 , then we can prepend symbols to w constructing a term uw of type T_6 , since for any sort T_i with $0 \leq i \leq 5$ there is a sequence of type $T_i \rightarrow T_6$: $\text{Rotate} :: T_5 \rightarrow T_6$, $\text{RotateCut} :: T_4 \rightarrow T_6$, $\text{RotateCut}c :: T_3 \rightarrow T_6$ (for some $c \in \Sigma_C$), $\text{RotateCut}cb :: T_2 \rightarrow T_6$ (for some $b \in \Sigma_B$ and some $c \in \Sigma_C$), and $\text{Begin} :: T_1 \rightarrow T_6$. ◀

By inspecting the typing the following characterization of terms of sort T_6 holds:

► **Lemma 28.** *Any term of sort T_6 is of one of the following forms:*

1. $\text{Rotate}w_E \text{Cut}w_D \text{Moveleft}cw_B \text{Wait}w_A \text{End}$ where $w_E \in \Sigma_E^*$, $w_D \in \Sigma_D^*$, $c \in \Sigma_C$, $w_B \in \Sigma_B^*$, and $w_A \in \Sigma_A^*$.
2. $\text{Rotate}w_E \text{Cut}w_D \text{Goright}w_B \text{Wait}w_A \text{End}$ where $w_E \in \Sigma_E^*$, $w_D \in \Sigma_D^*$, $w_B \in \Sigma_B^*$, and $w_A \in \Sigma_A^*$.
3. $\text{Rotate}w_E \text{Cut}w_D \text{Finish}w_A \text{End}$ where $w_E \in \Sigma_E^*$, $w_D \in \Sigma_D^*$, and $w_A \in \Sigma_A^*$.
4. $\text{Rotate}w_E \text{Cut}w_D \text{Guess}w_A \text{End}$ where $w_E \in \Sigma_E^*$, $w_D \in \Sigma_D^*$, and $w_A \in \Sigma_A^*$.
5. $\text{Rotate}w_E \text{Finish2}w_A \text{End}$ where $w_E \in \Sigma_E^*$ and $w_A \in \Sigma_A^*$.
6. $\text{Begin}w_A \text{End}$ where $w_A \in \Sigma_A^*$.
7. $\text{Rewrite}w_A \text{End}$ where $w_A \in \Sigma_A^*$.

► **Definition 29.** For a term $w :: T_6$ the mapping $\Phi_R(w) \in \mathbb{P}(\Sigma_A^*)$ is defined according to the cases of Lemma 28 as follows:

1. $\Phi_R(\text{Rotate}w_E \text{Cut}w_D \text{Moveleft}cw_B \text{Wait}w_A \text{End}) := \{ \langle w_D \rangle_A \langle w_B \rangle_A \langle w_E \rangle_A \langle c \rangle_A w_A \}$
2. $\Phi_R(\text{Rotate}w_E \text{Cut}w_D \text{Goright}w_B \text{Wait}w_A \text{End}) := \{ \langle w_D \rangle_A \langle w_B \rangle_A \langle w_E \rangle_A w_A \}$
3. $\Phi_R(\text{Rotate}w_E \text{Cut}w_D \text{Finish}w_A \text{End}) := \{ \langle w_D \rangle_A w_A \langle w_E \rangle_A \}$
4. $\Phi_R(\text{Rotate}w_E \text{Cut}w_D \text{Guess}w_A \text{End}) := \{ \langle w_D \rangle_A w'_A \langle w_E \rangle_A w''_A \mid \text{for all } w'_A w''_A = w_A \}$
5. $\Phi_R(\text{Rotate}w_E \text{Finish2}w_A \text{End}) := \{ \langle w_E \rangle_A w_A \}$
6. $\Phi_R(\text{Begin}w_A \text{End}) := \{ w_A \}$
7. $\Phi_R(\text{Rewrite}w_A \text{End}) := \{ w_A \}$

► **Lemma 30.** *Let $w, w' :: T_6$ with $w \rightarrow_{\text{rot}(\Sigma_A)} w'$. Then for any $u' \in \Phi_R(w')$ there exists $u \in \Phi_R(w)$ s.t. $u \sim u'$.*

Proof. We go through the cases for w according to Lemma 28

1. $w = \text{Rotate}_{w_E} \text{Cut}_{w_D} \text{Moveleft}_{c w_B} \text{Wait}_{w_A} \text{End}$. Rules (rotateF) and (rotateG) may be applied:
 - If (rotateF) is applied then let $w_D = w'_D d$.
We get $w' = \text{Rotate}_{w_E} \text{Cut}_{w'_D} \text{Moveleft}_{c \langle d \rangle_B} w_B \text{Wait}_{w_A} \text{End}$. Since $\Phi(w) = \Phi(w')$ the claim holds.
 - If (rotateG) is applied then $w_D = \varepsilon$ and $w' = \text{Rotate}_{w_E} e \text{Cut}_{\text{Goright}} w_B \text{Wait}_{w_A} \text{End}$. Since $\Phi(w) = \Phi(w')$ the claim holds.
2. $w = \text{Rotate}_{w_E} \text{Cut}_{w_D} \text{Goright}_{w_B} \text{Wait}_{w_A} \text{End}$. Rules (rotateH), (rotateI), and (rotateJ) may be applied:
 - If (rotateH) is applied then let $w_B = b w'_B$.
Then $w' = \text{Rotate}_{w_E} \text{Cut}_{w_D} \langle b \rangle_D \text{Goright}_{w'_B} \text{Wait}_{w_A} \text{End}$. Since $\Phi(w) = \Phi(w')$ the claim holds.
 - If (rotateI) is applied then $w_B = \varepsilon$ and let $w_A = a w'_A$.
Then $w' = \text{Rotate}_{w_E} \text{Cut}_{w_D} \text{Moveleft}_{\langle a \rangle_C} \text{Wait}_{w'_A} \text{End}$. Since $\Phi(w) = \Phi(w')$ the claim holds.
 - If (rotateJ) is applied then $w_A = w_B = \varepsilon$ Then $w' = \text{Rotate}_{w_E} \text{Cut}_{w_D} \text{Finish} \text{End}$. Since $\Phi(w) = \Phi(w')$ the claim holds.
3. $w = \text{Rotate}_{w_E} \text{Cut}_{w_D} \text{Finish}_{w_A} \text{End}$. Rules (rotateK) and (rotateL) may be applied:
 - If (rotateK) is applied then let $w_D = w'_D d$.
Then $w' = \text{Rotate}_{w_E} \text{Cut}_{w'_D} \text{Finish}_{\langle d \rangle_A} w_A \text{End}$. Since $\Phi(w) = \Phi(w')$ the claim holds.
 - If (rotateL) is applied then $w_D = \varepsilon$ and $w' = \text{Rotate}_{w_E} \text{Finish2}_{w_A} \text{End}$. Since $\Phi(w) = \{w_A \langle w_E \rangle_A\}$ and $\Phi(w') = \{\langle w_E \rangle_A w_A\}$ and $w_A \langle w_E \rangle_A \sim \langle w_E \rangle_A w_A$ the claim holds.
4. $w = \text{Rotate}_{w_E} \text{Cut}_{w_D} \text{Guess}_{w_A} \text{End}$. Rules (rotateC), (rotateD), and (rotateE) may be applied:
 - If (rotateC) is applied then let $w_A = a v_A$. Then $w' = \text{Rotate}_{w_E} \text{Cut}_{w_D} \langle a \rangle_D \text{Guess}_{v_A} \text{End}$. Since $\Phi(w) \supseteq \Phi(w') \neq \emptyset$ the claim holds.
 - If (rotateD) is applied then let $w_A = a v_A$.
Then $w' = \text{Rotate}_{w_E} \text{Cut}_{w_D} \text{Moveleft}_{\langle a \rangle_C} \text{Wait}_{v_A} \text{End}$. Since $\Phi(w) \supseteq \Phi(w') \neq \emptyset$ the claim holds.
 - If (rotateE) is applied then $w_A = \varepsilon$ and $w' = \text{Rotate}_{w_E} \text{Cut}_{w_D} \text{Finish} \text{End}$. In this case $\Phi(w) = \Phi(w')$ holds which shows the claim.
5. $w = \text{Rotate}_{w_E} \text{Finish2}_{w_A} \text{End}$. Rules (rotateM) and (rotateN) may be applied:
 - If (rotateM) is applied then let $w_E = w'_E e$ Then $w' = \text{Rotate}_{w'_E} \text{Finish2}_{\langle e \rangle_A} w_A \text{End}$. Since $\Phi(w) = \Phi(w')$ the claim holds.
 - If (rotateN) is applied then $w_E = \varepsilon$ and $w' = \text{Rewrite}_{w_A} \text{End}$. Since $\Phi(w) = \Phi(w')$ the claim holds.
6. $w = \text{Begin}_{w_A} \text{End}$. Rules (rotateA) or (rotateB) may be applied:
 - If (rotateA) is applied then $\Phi(w) = \{\varepsilon\} = \Phi(w')$ and thus the claim holds.
 - If (rotateB) is applied then let $w_A = a w'_A$. Then $w' = \text{Rotate}_{\text{Cut}} \langle a \rangle_D \text{Guess}_{w'_A}$ and since $\Phi(w) = \Phi(w')$ the claim holds.
7. $w = \text{Rewrite}_{w_A} \text{End}$. Then no rule can be applied. ◀

► **Proposition 31.** *If $w :: T_6$ admits an infinite $\rightarrow_{\text{rotate}(R)}$ -reduction, then there exists $u \in \Phi_R(w)$ s.t. $[u]$ admits an infinite $\circ \rightarrow$ -reduction.*

Proof. Let $w \rightarrow_{\text{rotate}(R)} w_1 \rightarrow_{\text{rotate}(R)} w_2 \rightarrow_{\text{rotate}(R)} \dots$ be an infinite $\rightarrow_{\text{rotate}(R)}$ -reduction. This sequence must have infinitely many steps using rules (rotateO), since the rewrite system $\text{rot}(\Sigma_A)$ is terminating (Lemma 23), i.e.

$$w = v_0 \xrightarrow{*}_{\text{rot}(\Sigma_A)} v_1 \xrightarrow{(\text{rotateO})} v_1' \xrightarrow{*}_{\text{rot}(\Sigma_A)} v_2 \xrightarrow{(\text{rotateO})} v_2' \dots$$

Lemma 30 shows that for any sub-sequence $v_i \xrightarrow{*}_{\text{rot}(\Sigma_A)} v_{i+1}$ and any $u_{i+1} \in \Phi_R(v_{i+1})$ there exists $u_i \in \Phi_R(v_i)$ with $u_i \sim u_{i+1}$. The steps $v_i \xrightarrow{(\text{rotateO})} v_i'$ are of the form $\text{Rewrite} \ell w_A \rightarrow \text{Begin} r w_A$ for some rule $(\ell \rightarrow r) \in R$. Let $u_i := \ell w_A$ and $u_i' := r w_A$. Since $\Phi_R(v_i) = \{u_i\}$ and $\Phi_R(v_i') = \{u_i'\}$ and since $u_i \rightarrow_R u_i'$ we also have $[u_i] \circ \rightarrow_R [u_i']$. Thus we can construct the infinite sequence $u_0 \sim u_1 \circ \rightarrow_R u_1' \sim u_2 \circ \rightarrow_R \dots$ which shows the claim. \blacktriangleleft

► **Theorem 32.** *The transformation rotate is sound and complete.*

Proof. Soundness is shown in Theorem 26, completeness follows by type introduction, Lemma 27, and Proposition 31. \blacktriangleleft

A.3 Soundness and Completeness of Shift

Let us write $\widetilde{\Sigma}_A := \Sigma_A \cup \Sigma_B \cup \Sigma_C \cup \{\text{B, E, W, V, M, L, R, D}\}$.

► **Lemma 33.** *If $u \curvearrowright^{<\text{len}(R)} v \rightarrow_R w$, then $\text{B}u\text{E} \xrightarrow{+}_{\text{shift}(R)} \text{B}w\text{E}$.*

Proof. Let $\text{len}(R) = m + 1$, $u = a_1 \dots a_n$ and $u \curvearrowright^k a_{k+1} \dots a_n a_1 \dots a_k = v$, and let $v = a'_1 \dots a'_n$ and $w = a'_1 \dots a'_i a''_1 \dots a''_r a'_{i+j} \dots a'_n$, i.e. the applied rewrite rule is $a'_{i+1} \dots a'_{i+j-1} \rightarrow a''_1 \dots a''_r$. Then the following rewrite sequence using $\text{shift}(R)$ can be constructed:

$$\begin{aligned} & \text{B}a_1 \dots a_n \text{E} \xrightarrow{(\text{shiftA})} \text{W} \text{M}^m \text{V} a_1 \dots a_n \text{E} \xrightarrow{(\text{shiftB})^*} \text{W} \text{M}^k \text{V} a_1 \dots a_n \text{E} \\ & \xrightarrow{(\text{shiftC})} \text{W} \text{M}^{k-1} \text{V} (a_1)_B a_2 \dots a_n \text{E} \xrightarrow{(\text{shiftD})^*} \text{W} \text{M}^{k-1} \text{V} a_2 \dots a_n (a_1)_B \text{E} \\ & \xrightarrow{(\text{shiftE})} \text{W} \text{M}^{k-1} \text{V} a_2 \dots a_n a_1 \text{E} \dots \xrightarrow{(\text{shiftE})} \text{W} \text{V} a_{k+1} \dots a_n a_1 \dots a_k \text{E} \\ & \xrightarrow{(\text{shiftF})} \text{R} \text{L} a_{k+1} \dots a_n a_1 \dots a_k \text{E} = \text{R} \text{L} a'_1 \dots a'_n \text{E} \xrightarrow{(\text{shiftG})^*} \text{R} (a'_1)_C \dots (a'_i)_C \text{L} a'_{i+1} \dots a'_n \text{E} \\ & \xrightarrow{(\text{shiftH})} \text{R} (a'_1)_C \dots (a'_i)_C \text{D} a''_1 \dots a''_r a'_{i+j} \dots a'_n \text{E} \xrightarrow{(\text{shiftI})^*} \text{R} \text{D} a'_1 \dots a'_i a''_1 \dots a''_r a'_{i+j} \dots a'_n \text{E} \\ & \xrightarrow{(\text{shiftJ})^*} \text{B} a'_1 \dots a'_i a''_1 \dots a''_r a'_{i+j} \dots a'_n \text{E} \end{aligned} \quad \blacktriangleleft$$

Thus by Proposition 12 we have soundness of shift . We prove completeness by using type introduction. Let T_0, \dots, T_n be sorts. Let c be a constant of sort T_0 . We interpret the symbols of $\widetilde{\Sigma}_A$ as unary function symbols with the following types:

$$\begin{array}{llll} a & :: T_1 \rightarrow T_1 & \text{for all } a \in \Sigma_A & \text{E} & :: T_0 \rightarrow T_1 & \text{W} & :: T_2 \rightarrow T_3 & \text{D} & :: T_1 \rightarrow T_4 \\ b & :: T_1 \rightarrow T_1 & \text{for all } b \in \Sigma_B & \text{V} & :: T_1 \rightarrow T_2 & \text{B} & :: T_1 \rightarrow T_3 & \text{R} & :: T_4 \rightarrow T_3 \\ c & :: T_4 \rightarrow T_4 & \text{for all } c \in \Sigma_C & \text{M} & :: T_2 \rightarrow T_2 & \text{L} & :: T_1 \rightarrow T_4 \end{array}$$

We verify that the rules of $\text{shift}(R)$ are well-typed: Rules (shiftD) and (shiftE) rewrite terms of sort T_1 , rules (shiftB) and (shiftC) rewrite terms of sort T_2 , rules (shiftA), (shiftF), (shiftJ) rewrite terms of sort T_3 , and rules (shiftG), (shiftH), (shiftI) rewrite terms of sort T_4 .

► **Lemma 34.** *Let $w \in \widetilde{\Sigma}_A^*$ be well-typed, s.t. w admits an infinite reduction w.r.t. $\rightarrow_{\text{shift}(R)}$. Then there exists a well-typed term $w' \in \widetilde{\Sigma}_A^*$ of sort T_3 which admits an infinite reduction.*

Proof. If w is not of sort T_4 , then we can prepend symbols to w constructing a term uw of type T_3 , since for any sort $T_i \in \{T_0, T_1, T_2, T_4\}$ there is a sequence of type $T_i \rightarrow T_3$: $\text{R} :: T_4 \rightarrow T_3$, $\text{W} :: T_2 \rightarrow T_3$, $\text{B} :: T_1 \rightarrow T_3$, and $\text{BE} :: T_0 \rightarrow T_3$. \blacktriangleleft

We use string notation for terms and write E instead of Ec

► **Lemma 35.** *Any well-typed term of type T_3 is of one of the following forms:*

- WM^iVwE where $i \geq 0$ and $w \in (\Sigma_A \cup \Sigma_B)^*$
- BwE where $w \in (\Sigma_A \cup \Sigma_B)^*$
- Rw_cLwE where $w_c \in \Sigma_C^*$ and $w \in (\Sigma_A \cup \Sigma_B)^*$
- Rw_cDwE where $w_c \in \Sigma_C^*$ and $w \in (\Sigma_A \cup \Sigma_B)^*$

For a string $w \in (\Sigma_A \cup \Sigma_B)^*$ let $\pi_A(w)$ be the string w where all symbols $b \in \Sigma_B$ are removed and let $\bar{\pi}_B(w)$ be the reversed string of w' where w' is w where all symbols $a \in \Sigma_A$ are removed.

► **Definition 36.** For a term $w :: T_3$ the mapping $\Phi_S(w) \in \Sigma_A^*$ is defined according to the cases of Lemma 35 as follows:

1. $\Phi_S(WM^iVwE) := \pi_A(w)\bar{\pi}_B(w)$.
2. $\Phi_S(BwE) := \pi_A(w)\bar{\pi}_B(w)$.
3. $\Phi_S(Rw_cLwE) := (w_c)_A \pi_A(w)\bar{\pi}_B(w)$.
4. $\Phi_S(Rw_cDwE) := (w_c)_A \pi_A(w)\bar{\pi}_B(w)$.

► **Lemma 37.** *If $u :: T_3 \rightarrow_{\text{shift}(R)} v :: T_3$, then $[\Phi_S(u)] \circ \rightarrow_R^* [\Phi_S(v)]$.*

Proof. We go through the cases of Lemma 35 and inspect all applicable rewrite rules:

- $u = WM^iVwE$. Then there are the following cases: If rule (shiftB), (shiftD), (shiftE), or (shiftF) is applied then $\Phi_S(u) = \Phi_S(v)$ and if rule (shiftC) is applied, then $\Phi_S(u) = aw_1w_2$ (with $w = aw'$ and $\pi_A(w) = aw_1$ and $\bar{\pi}_B(w) = w_2$) and $\Phi_S(v) = w_1w_2a$ and thus $\Phi_S(u) \sim \Phi_S(v)$.
- $u = BwE$. Rules (shiftA), (shiftD), or (shiftE) may be applied and in all cases $\Phi_S(u) = \Phi_S(v)$ holds.
- $u = Rw_cLwE$. If rule (shiftD), (shiftE), or (shiftG) is applied, then $\Phi_S(u) = \Phi_S(v)$. If rule (shiftH) is applied then $\Phi_S(u) \rightarrow_R \Phi_S(v)$
- $u = Rw_cDwE$. Rules (shiftD), (shiftE), and (shiftI) may be applied and in all cases $\Phi_S(u) = \Phi_S(v)$ holds. ◀

► **Proposition 38.** *Let $u :: T_3$ s.t. u admits an infinite $\rightarrow_{\text{shift}(R)}$ -reduction. Then $[u]$ admits and infinite $\circ \rightarrow_R$ reduction.*

Proof. The previous lemma shows that given the infinite reduction for u it is possible to construct an infinite reduction of $\circ \rightarrow_R^*$ -steps. Since applications of rule (shiftH) are translated into $\circ \rightarrow_R$ steps, it remains to show that the infinite reduction of u has infinitely many applications of rule (shiftH). This holds, since the following polynomial interpretation σ shows that the rewrite system consisting of all rules of $\text{shift}(R)$ except for rule (shiftH) is terminating: Let $N = \max\{1, \text{len}(R)\}$ and

$$\begin{array}{lll} \sigma(a) = \lambda x.4x + 4 \text{ for all } a \in \Sigma_A & \sigma(D) = \lambda x.2^N x + 2^{N+1} - 1 & \sigma(M) = \lambda x.2x + 1 \\ \sigma(b) = \lambda x.8x + 1 \text{ for all } b \in \Sigma_B & \sigma(R) = \lambda x.2x + 1 & \sigma(L) = \lambda x.x \\ \sigma(c) = \lambda x.4x \text{ for all } c \in \Sigma_C & \sigma(E) = \lambda x.7x + 1 & \sigma(W) = \lambda x.2x + 2 \\ \sigma(B) = \lambda x.2^N x + 2^{N+1} - 1 & \sigma(V) = \lambda x.x & \end{array}$$

Then $\sigma(\ell) > \sigma(r)$ for all $(\ell \rightarrow r) \in \left(\text{shift}(R) \setminus \{\ell \xrightarrow{\text{(shiftH)}} r\} \right)$:

- rule (shiftA): $\lambda x.2^N x + 2^{N+1} - 1 > \lambda x.2^N x + 2^N$,
- rule (shiftB): $\lambda x.2x + 1 > \lambda x.x$,
- rule (shiftC): $\lambda x.8x + 9 > \lambda x.8x + 1$,
- rule (shiftD): $\lambda x.32x + 33 > \lambda x.32x + 8$,
- rule (shiftE): $\lambda x.56x + 9 > \lambda x.28x + 8$,
- rule (shiftF): $\lambda x.2x + 2 > \lambda x.2x + 1$,
- rule (shiftG): $\lambda x.4x + 4 > \lambda x.4x$,
- rule (shiftI): $\lambda x.2^{N+2}x + 3 \cdot 2^{N+1} + 2^{N+1} - 4 > \lambda x.2^{N+2}x + 3 \cdot 2^{N+1} - 1$,
- rule (shiftJ): $\lambda x.2^{N+1}x + 2^{N+2} - 1 > \lambda x.2^N x + 2^{N+1} - 1$.

Thus termination of $\left(\text{shift}(R) \setminus \{ \ell \xrightarrow{(\text{shiftH})} r \} \right)$ has been shown. ◀

► **Theorem 39.** *The transformation shift is sound and complete.*

Proof. Soundness follows by Lemma 33 and Proposition 12. Completeness follows by type introduction, Proposition 38 and Lemma 34. ◀